1. $\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$

Sol: take $x = \sin t \Rightarrow dx = \cos t\,dt$.

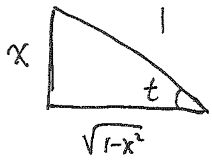
$$\Rightarrow \int \frac{1}{x^2\sqrt{1-x^2}}\,dx = \int \frac{1}{\sin^2 t \cdot \cos t}\cdot \cos t\,dt$$

$$= \int \frac{1}{\sin^2 t}\,dt$$

$$= -\cot(t) + C$$

$\left((\tan t)' = \sec^2 t \Rightarrow \int \sec^2 t\,dt = \tan t + C \right.$

$\left. (\cot t)' = -\csc^2 t \quad -\int \csc^2 t\,dt = \cot t + C \right)$



$$\cot(t) = \frac{\sqrt{1-x^2}}{x}$$

$$= -\frac{\sqrt{1-x^2}}{x} + C$$

2. $\int \frac{1}{1+x+x^2}\,dx$

Sol: Recall $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

$$1+x+x^2 = (x+\tfrac{1}{2})^2 + \tfrac{3}{4} = \tfrac{3}{4}\left[1 + \tfrac{4}{3}(x+\tfrac{1}{2})^2\right]$$

so take $t = \frac{2}{\sqrt{3}}(x+\frac{1}{2})$ $\quad dt = \frac{2}{\sqrt{3}}\,dx \Rightarrow dx = \frac{\sqrt{3}}{2}\,dt$

$$\int \frac{1}{1+x+x^2}\,dx = \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{1}{1+t^2}\,dt$$

$$= \frac{2\sqrt{3}}{3}\arctan t + C$$

$$= \frac{2\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}(x+\tfrac{1}{2})\right) + C$$

3. $\int \frac{4-2x}{\sqrt{3-2x-x^2}}\,dx$.

Sol: First set $t = 3-2x-x^2$ so $dt = (-2x-2)\,dx$.

$$\Rightarrow \int \frac{4-2x}{\sqrt{3-2x-x^2}}\,dx = \int \frac{-2x-2}{\sqrt{3-2x-x^2}}\,dx + 6\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$$

Recall $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

$$= \int \frac{d(3-2x-x^2)}{\sqrt{3-2x-x^2}} + 6\int \frac{1}{2\sqrt{1-(\frac{x+1}{2})^2}}\,dx$$

$3-2x-x^2 = 4-(x+1)^2$

$= 4\left[1-(\frac{x+1}{2})^2\right]$

$$= 2\sqrt{3-2x-x^2} + 6\arcsin\left(\frac{1+x}{2}\right) + C$$

take $u = \frac{1+x}{2}$ $\quad du = \frac{1}{2}dx \Rightarrow dx = 2du$

4. $\int \frac{1+x+x^2}{1-x^4}\,dx$

Sol: $1-x^4 = (1-x)(1+x)(1+x^2)$

so the partial fraction would be:

$$\frac{1+x+x^2}{1-x^4} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2} = \frac{A(1+x)(1+x^2)+B(1-x)(1+x^2)+(Cx+D)(1-x^2)}{(1-x)(1+x)(1+x^2)}$$

$$\begin{aligned} x^0&: A+B+D=1 \\ x^1&: A-B+C=1 \\ x^2&: A+B-D=1 \\ x^3&: A-B-C=0 \end{aligned} \quad\Rightarrow\quad \begin{aligned} A&=\tfrac{3}{4} \\ B&=\tfrac{1}{4} \\ C&=\tfrac{1}{2} \\ D&=0 \end{aligned}$$

$$\Rightarrow \int \frac{1+x+x^2}{1-x^4}\,dx = \frac{3}{4}\int \frac{1}{1-x}\,dx + \frac{1}{4}\int \frac{1}{1+x}\,dx + \frac{1}{2}\int \frac{x}{1+x^2}\,dx$$

$$= -\frac{3}{4}\ln|x-1| + \frac{1}{4}\ln|1+x| + \frac{1}{4}\ln(1+x^2) + C$$

5. Method-1: denote $f(x) = (x^2-1)^n = \sum_{k=0}^{n} C_k^n x^{2k}(-1)^{n-k}$

the constant wouldn't affect the result so we consider the x^{2k}

$f^{(n)}(x)$ means do the differential n times for x^{2k}. so it would be:

$(x^{2k})^{(n)} = 2k(2k-1)\cdots(2k-n+1)\,x^{2k-n}$

and $x^{2k-n}\cdot x^{n-1} = x^{2k-1}$ $(k\geq 1)$ (for $k=0$ is the constant term, it would be 0 after differential)

$2k-1$ always be odd so x^{2k-1} is odd function.

then $\int_{-1}^{1} f^{(n)}(x)\cdot x^{n-1}\,dx \Rightarrow C_k\int_{-1}^{1} x^{2k-1}\,dx = 0$, that's what we mean C_k the constant would not affect the result..

So $\int_{-1}^{1} f^{(n)}(x)\cdot x^{n-1}\,dx = 0 \quad \forall n\geq 1$.

Remark: I find there is something wrong for the mathematical induction, so I just use this one. Sorry for the mistake.